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Geometric means on gyrogroups

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1 Introdiction

Abraham Albert Ungar initiated the theory of gyrogroups in 1989 [1] associated with the study of Einstein's velocity addition in the theory of special relativity. It is the study of analytic hyperbolic geometry. A gyrogroup has a weak associativity. It is a generalization of a group. The set of all positive invertible elements in a unital C^*-algebra is an example of a gyrogroup. It is difficult to give an appropriate definition for "the geometric mean" of more than two points on a gyrovector space because of nonassociativity and noncommutativity of the operation. So we define a geometric mean for the Einstein gyrovector space.

2 Einstein gyrovector space

Einstainian velocities with the Einstain's velocity addition based on the special theory of relativity is a gyrocommutative gyrogroup. Let $\mathbb{V}$ be a real Hilbert space. Let $\mathbb{V}_1$ be an open unit ball of $\mathbb{V}$, that is,

$$\mathbb{V}_1=\{\mathbf{v}\in\mathbb{V}:\|\mathbf{v}\|<1\}.$$

The Einstein addition $\oplus_E$ on $\mathbb{V}_1$ is a binary operation on $\mathbb{V}_1$ given by the equation

$$\mathbf{a}\oplus_E\mathbf{b}=\frac{1}{1+\mathbf{a}\cdot\mathbf{b}}\left\{\mathbf{a}+\frac{1}{\gamma_{\mathbf{a}}}\mathbf{b}+\frac{\gamma_{\mathbf{a}}}{1+\gamma_{\mathbf{a}}}(\mathbf{a}\cdot\mathbf{b})\mathbf{a}\right\},$$

where $\gamma_{\mathbf{u}}$ is the Lorentz gamma factor defined by

$$\gamma_{\mathbf{u}}=\frac{1}{\sqrt{1-\|\mathbf{u}\|^2}}.$$

These $\cdot$ and $\|\cdot\|$ denote the usual inner product and the norm of $\mathbb{V}$ respectively. Einstein scalar multiplication $\otimes_E$ is given by the form

$$\begin{aligned}r\otimes_E\mathbf{a}&=\frac{(1+\|\mathbf{a}\|)^r-(1-\|\mathbf{a}\|)^r}{(1+\|\mathbf{a}\|)^r+(1-\|\mathbf{a}\|)^r}\frac{\mathbf{a}}{\|\mathbf{a}\|}\\&=\tanh(r\tanh^{-1}\|\mathbf{a}\|)\frac{\mathbf{a}}{\|\mathbf{a}\|},\end{aligned}$$

where $r \in \mathbb{R}, \mathbf{a} \in \mathbb{V}_1 \backslash \{\mathbf{0}\}$ and $r \otimes_E \mathbf{0} = \mathbf{0}$. By Theorem 6.84 in [2], $(\mathbb{V}_1, \oplus_E, \otimes_E)$ is defined by a gyrovector space.

We define the set $\|\mathbb{V}_1\| = \{\pm\|\mathbf{v}\| : \mathbf{v} \in \mathbb{V}_1\} \subset \mathbb{R}$, which coincides with the open interval $(-1,\ 1)$. $\|\mathbb{V}_1\|$ admits addition $\oplus'_E$ and scalar multiplication $\otimes'_E$ given by the following:

$$a \oplus'_E b = \frac{a+b}{1+ab}$$
$$r \otimes'_E a = \tanh(r \tanh^{-1} a)$$

where $a, b \in \|\mathbb{V}_1\|$ and $r \in \mathbb{R}$. These two operator is induced by $\oplus_E$ and $\otimes_E$. $(\|\mathbb{V}_1\|, \oplus'_E, \otimes'_E)$ is a real one dimensional space.

3 The metric space on $(\mathbb{V}_1, \oplus_E, \otimes_E)$

The gyrometric is defined by

$$d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} \ominus_E \mathbf{b}\| \in |\mathbb{V}_1\|,$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{V}_1$ and $\mathbf{a} \ominus_E \mathbf{b} = \mathbf{a} \oplus_E (-\mathbf{b})$. The gyrometric is not a metric. It satisfies the following properties:

(1) $d(\mathbf{a}, \mathbf{b}) \geq 0$ for every $\mathbf{a}, \mathbf{b} \in \mathbb{V}_1$, $d(\mathbf{a}, \mathbf{b}) = 0 \Leftrightarrow \mathbf{a} = \mathbf{b}$.

(2) $d(\mathbf{a}, \mathbf{b}) = d(\mathbf{b}, \mathbf{a})$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{V}_1$.

(3) The gyrotriangle inequality:

$$d(\mathbf{a}, \mathbf{b}) \leq d(\mathbf{a}, \mathbf{c}) \oplus'_E d(\mathbf{c}, \mathbf{b})$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_1$.

We define a metric on $\mathbb{V}_1$ induced the gyrometric; $f : \|\mathbb{V}_1\| \to \mathbb{R}$ is $f(x) = \tanh^{-1}(x)$. Then f is monotonic and satisfies the following properties:

(F1) $f(a \oplus'_E b) = f(a) + f(b)$ for all $a, b \in \|\mathbb{V}_1\|$

(F2) $f(r \otimes'_E a) = rf(a)$ for all $a \in \|\mathbb{V}_1\|$ and $r \in \mathbb{R}$.

We define $\delta(\mathbf{a}, \mathbf{b}) = f(d(\mathbf{a}, \mathbf{b}))$, where $\mathbf{a}, \mathbf{b} \in \mathbb{V}_1$.

Proposition 1. *The map δ give a metric on $\mathbb{V}_1$; $(\mathbb{V}_1, \delta)$ is a complete metric space.*

4 Gyromidpoints and gyrocentroids

Ungar defined the gyromidpoint $\mathbf{P}^m_{\mathbf{ab}}$ of two elements $\mathbf{a}, \mathbf{b} \in \mathbb{V}_1$ given by

$$\mathbf{P}^m_{\mathbf{ab}} = \frac{\gamma_{\mathbf{a}}\mathbf{a} + \gamma_{\mathbf{b}}\mathbf{b}}{\gamma_{\mathbf{a}} + \gamma_{\mathbf{b}}},$$

By a natural extension, Ungar [2] define the gyrocentroid $\mathbb{C}^m_{\mathbf{abc}}$ of three elements $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_1$ written by

$$\mathbb{C}^m_{\mathbf{abc}} = \frac{\gamma_{\mathbf{a}}\mathbf{a} + \gamma_{\mathbf{b}}\mathbf{b} + \gamma_{\mathbf{c}}\mathbf{c}}{\gamma_{\mathbf{a}} + \gamma_{\mathbf{b}} + \gamma_{\mathbf{c}}}.$$

The gyromidpoint $\mathbf{P}^m_{\mathbf{ab}}$ satisfies some desirable properties one would expect for means, for example the permutation invariance and the left gyrotranslation invariance which is given by

$$\mathbf{x} \oplus_E \mathbf{P}^m_{\mathbf{ab}} = \mathbf{P}^m_{(\mathbf{x}\oplus_E\mathbf{a})(\mathbf{x}\oplus_E\mathbf{b})}.$$

But the gyrocentroid does not satisfy the left gyrotranslation invariance. In the case of the three points $\mathbf{0}, \mathbf{0}, \mathbf{c}$, $\mathbb{C}^m_{\mathbf{abc}} \neq \frac{1}{3} \otimes_E \mathbf{c}$ by the simple calculation. In this paper we will give a definition of a geometric mean alternatively to remove these difficulties.

5 Definition of the geometric mean

We define the geometric mean $G(\mathbf{a}, \mathbf{b}, \mathbf{c})$ of three elements $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_1$ as in the following. Starting from $\mathbf{a}_0 = \mathbf{a}, \mathbf{b}_0 = \mathbf{b}, \mathbf{c}_0 = \mathbf{c}$, we define $\mathbf{a}_m, \mathbf{b}_m, \mathbf{c}_m$ by induction on m. Suppose that $\mathbf{a}_{m-1}, \mathbf{b}_{m-1}, \mathbf{c}_{m-1}$ are defined. Then

$$\mathbf{a}_m = \mathbf{a}_{m-1}\#\mathbf{b}_{m-1}, \mathbf{b}_m = \mathbf{b}_{m-1}\#\mathbf{c}_{m-1}, \mathbf{c}_m = \mathbf{c}_{m-1}\#\mathbf{a}_{m-1}$$

where $\mathbf{x}\#\mathbf{y}$ is the gyromidpoint of $\mathbf{x}$ and $\mathbf{y}$. Then $\lim_{m\to\infty} \mathbf{a}_m, \lim_{m\to\infty} \mathbf{b}_m, \lim_{m\to\infty} \mathbf{c}_m$ exist and they coincide with each other. Define the common limit as M_∞. We define $G(\mathbf{a}, \mathbf{b}, \mathbf{c}) = M_\infty$. $G(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is permutation invariant. By a simple calculation, $G(\mathbf{0}, \mathbf{0}, \mathbf{c}) = \frac{1}{3} \otimes_E \mathbf{c}$ holds. We define the geometric mean for any finite number of elements as follows. Let Δ_n be a n-points set of $\mathbb{V}_1$. We define the geometric mean $G(\Delta_n)$ by induction of the number of elements n by generalizing the way as above.

Definition 1. *(1) Let $\Delta_2 = \{\mathbf{a}_1, \mathbf{a}_2\} \subset \mathbb{V}_1$. We define $G(\Delta_2) = \mathbf{a}_1\#\mathbf{a}_2$.*

(2) Suppose that we have defined the geometric mean $G(\Delta_n)$ for any Δ_n. Let $\Delta_{n+1} = \{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_{n+1}\} \subset \mathbb{V}_1$. Put $\mathbf{a}^0_i = \mathbf{a}_i$ for $i = 1, 2, \cdots, n+1$. For a positive integer m, we define $\mathbf{a}^m_i$ for $i = 1, 2, \cdots, n+1$ by induction on m as follows. Suppose that $\mathbf{a}^{m-1}_i$ for $i = 1, 2, \cdots, n+1$ is defined, put

$$\mathbf{a}^m_i = G(\{\mathbf{a}^{m-1}_1, \mathbf{a}^{m-1}_2, \cdots, \mathbf{a}^{m-1}_{i-1}, \mathbf{a}^{m-1}_{i+1}, \cdots, \mathbf{a}^{m-1}_{n+1}\})$$

for $i = 1, 2, \cdots, n+1$. The point $\mathbf{a}^m_i$ is well defined since we suppose that $G(\Delta_n)$ is defined for an n-point set Δ_n.

Put $\Delta^m_{n+1} = \{\mathbf{a}^m_1, \mathbf{a}^m_2, \cdots, \mathbf{a}^m_{n+1}\}$. Then the limit $\lim_{m\to\infty} \mathbf{a}^m_i$ exists for each $i = 1, 2, \cdots, n+1$ and they coincide with each other. Define the common limit by M_∞. We define $G(\Delta_{n+1})$ to be M_∞.

This definition is a modification of the definition of the geometric mean for a positive definite matrices by Ando, Chi-Kwong Li and Mathias[5]

6 A proof of the existence of and the coincidence of $\lim_{m\to\infty} \mathbf{a}_i^m$

To prove the existence of $\lim_{m\to\infty} \mathbf{a}_i^m$ and the coincidenceness of each other, we need some preperations. We define a gyroline and a gyrosegment in the Einstein gyrovector space.

Definition 2. *Let* $\mathbf{a}, \mathbf{b}$ *be elements of* $\mathbb{V}_1$. *The gyroline* $L(\mathbf{a}, \mathbf{b}) = \{\mathbf{a} \oplus_E t \otimes_E (\ominus_E \mathbf{a} \oplus_E \mathbf{b}) \ : \ t \in \mathbb{R}\}$. *The gyrosegment* $S(\mathbf{a}, \mathbf{b}) = \{\mathbf{a} \oplus_E t \otimes_E (\ominus_E \mathbf{a} \oplus_E \mathbf{b}) \ : \ 0 \leq t \leq 1\}$.

$\mathbf{a}\#_t\mathbf{b} = \mathbf{a} \oplus_E t \otimes_E (\ominus_E \mathbf{a} \oplus_E \mathbf{b})$ is called a gyro t-point on a gyroline or gyrosegment. If $t = \frac{1}{2}$ then it is the gyromidpoint, that is

$$\mathbf{a}\#\mathbf{b} = \frac{1}{2} \otimes_E (\mathbf{a} \boxplus_E \mathbf{b}),$$

where $\boxplus_E$ is coaddition on $(\mathbb{V}_1, \oplus_E, \otimes_E)$ defined by the following;

$$\mathbf{a} \boxplus_E \mathbf{b} = \frac{\gamma_{\mathbf{a}} + \gamma_{\mathbf{b}}}{{\gamma_{\mathbf{a}}}^2 + {\gamma_{\mathbf{b}}}^2 + \gamma_{\mathbf{a}}\gamma_{\mathbf{b}}(\mathbf{a} \cdot \mathbf{b}) - 1}(\gamma_{\mathbf{a}}\mathbf{a} + \gamma_{\mathbf{b}}\mathbf{b}).$$

Proposition 2. *Coaddition* $\boxplus_E$ *is commutative, thus* $\mathbf{a}\#\mathbf{b} = \mathbf{b}\#\mathbf{a}$.

By Proposition2, the gyromidpoint is permutation invariant.
We define a gyroconvex set and a gyroconvex hull in the Einstein gyrovector space.

Definition 3. *A subset* C *of* $\mathbb{V}_1$ *is gyroconvex set if for any* $\mathbf{a}, \mathbf{b} \in C$, $S(\mathbf{a}, \mathbf{b}) \subset C$.

Definition 4. $X \subset \mathbb{V}_1$. *A gyroconvex hull of* X *is defined by:*

$$conv(X) = \cap\{C \subset \mathbb{V}_1 \ : \ X \subset C \text{ and } C \text{ is convex}\}.$$

These definitions are modifications of definitions of the geometric mean for a positive definite matrices by Bhatia and Holbrock[4]. By a simple calculation, we have the following property.

Proposition 3. *The set* C_m *for every* $m \in \mathbb{N} \cap \{0\}$ *is defined by induction. Let* $C_0 = X$. *If* C_{m-1} *is defined, then put* $C_m = \cup_{\mathbf{a},\mathbf{b} \in C_{m-1}} S(\mathbf{a}, \mathbf{b})$. *A gyroconvex hull of* X *can be written by*

$$conv(X) = \bigcup_{m=0}^{\infty} C_m.$$

We show an inequality which is related gyromidpoints. It plays a crucial role in the proof of the convergence of the sequence $\{\mathbf{a}_i^m\}$ we have defined before.

Theorem 1. *For any* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_1$ *we have*

$$d(\mathbf{a}\#\mathbf{b}, \mathbf{a}\#\mathbf{c}) \leq \frac{1}{2} \otimes'_E d(\mathbf{b}, \mathbf{c}).$$

This theorem is proved by a simple calculation. By applying gamma identities, we have

$$\begin{aligned}&\left(\frac{1}{2}\otimes_E d(\mathbf{b},\mathbf{c})\right)^2 - d^2(\mathbf{a}\#\mathbf{b},\mathbf{a}\#\mathbf{c})\\ &= \frac{2\left\{2\gamma_{\mathbf{a}\ominus_{\mathbf{E}}\mathbf{b}}\gamma_{\mathbf{b}\ominus_{\mathbf{E}}\mathbf{c}}\gamma_{\mathbf{c}\ominus_{\mathbf{E}}\mathbf{a}} - ({\gamma_{\mathbf{a}\ominus_{\mathbf{E}}\mathbf{b}}}^2 + {\gamma_{\mathbf{b}\ominus_{\mathbf{E}}\mathbf{c}}}^2 + {\gamma_{\mathbf{c}\ominus_{\mathbf{E}}\mathbf{a}}}^2) + 1\right\}}{(1+\gamma_{\mathbf{b}\ominus_{\mathbf{E}}\mathbf{c}})(1+\gamma_{\mathbf{a}\ominus_{\mathbf{E}}\mathbf{b}}+\gamma_{\mathbf{b}\ominus_{\mathbf{E}}\mathbf{c}}+\gamma_{\mathbf{c}\ominus_{\mathbf{E}}\mathbf{a}})^2}.\end{aligned} \tag{6.1}$$

Applying the left gyrotranslation of $\mathbf{c}$, put $\mathbf{A} = \ominus_E \mathbf{c} \oplus_E \mathbf{a}, \mathbf{B} = \ominus_E \mathbf{c} \oplus_E \mathbf{b}$, we calculate the numerator of (6.1),

$$\begin{aligned}&2\gamma_{\mathbf{A}\ominus_{\mathbf{E}}\mathbf{B}}\gamma_{\mathbf{A}}\gamma_{\mathbf{B}} - ({\gamma_{\mathbf{A}\ominus_{\mathbf{E}}\mathbf{B}}}^2 + {\gamma_{\mathbf{A}}}^2 + {\gamma_{\mathbf{B}}}^2) + 1\\ &= \frac{\|\mathbf{A}\|^2\|\mathbf{B}\|^2 - (\mathbf{A}\cdot\mathbf{B})^2}{(1-\|\mathbf{A}\|^2)(1-\|\mathbf{B}\|^2)} \geq 0.\end{aligned}$$

Note: I would like to thank Professor Akinari Hoshi for his calculation about the numerator by computer. By his calculation I convinced that the numerator is greater than or equal to 0. Finally I succeeded to prove it. Theorem 1 is followed by Corollary 1.

Corollary 1. $\delta(\mathbf{a}\#\mathbf{b},\mathbf{a}\#\mathbf{c}) \leq \frac{1}{2}\delta(\mathbf{b},\mathbf{c})$ *and hence*

$$\delta(\mathbf{a}\#\mathbf{b},\mathbf{c}\#\mathbf{d}) \leq \frac{1}{2}\delta(\mathbf{b},\mathbf{d}) + \frac{1}{2}\delta(\mathbf{a},\mathbf{c})$$

Moreover, since $g(t) = \delta(\mathbf{a}\#_t\mathbf{b},\mathbf{c}\#_t\mathbf{d})$ *is continuous, then* g *is convex, i.e.,*

$$\delta(\mathbf{a}\#_t\mathbf{b},\mathbf{c}\#_t\mathbf{d}) \leq (1-t)\delta(\mathbf{a},\mathbf{c}) + t\delta(\mathbf{b},\mathbf{d})$$

especially,

$$\delta(\mathbf{a}\#_t\mathbf{b},\mathbf{a}\#_t\mathbf{c}) \leq t\delta(\mathbf{b},\mathbf{c})$$

We define $diam(X) = \sup\{\delta(\mathbf{x},\mathbf{y}) : \mathbf{x},\mathbf{y} \in X\}$. We have the following properties. By Corollary 1, we have the following

Proposition 4. *If* $diam(\{\mathbf{x}_0,\mathbf{y}_0,\mathbf{x}_1,\mathbf{y}_1\}) \leq M$*, then for arbitrary* $\mathbf{x} \in S(\mathbf{x}_0,\mathbf{x}_1)$ *and* $\mathbf{y} \in S(\mathbf{y}_0,\mathbf{y}_1)$, $\delta(\mathbf{x},\mathbf{y}) \leq M$ *holds.*

Proposition 5 is proved by applying Proposition 3 and Proposition 4.

Proposition 5. *Let* X *be a subset of* $\mathbb{V}_1$

$$diam(conv(X)) = diam(X).$$

Considering the geometric mean of three elements, by Corollary 1 and Proposition 5, we have

$$diam(conv(\Delta_3^m)) \leq \frac{1}{2}diam(conv(\Delta_3^{m-1})).$$

Since $(\mathbb{V}_1, \delta)$ is a complete, there exists $M_\infty \in \mathbb{V}_1$ such that $\bigcap_{m=0}^{\infty} conv(\Delta_3^m) = \{M_\infty\}$ by the Cantor's intersection principle. So the limit of $\mathbf{a}_i^m$ exists for $i = 1, 2, 3$ and they coincide with each other.

7 Properties of the geometric mean

To prove the existence of the geometric mean, we assume the following inequality by induction.

Theorem 2. *For any sets of n elements in $\mathbb{V}_1$, $D_n = \{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}$, $D'_n = \{\mathbf{a'}_1, \mathbf{a'}_2, \cdots, \mathbf{a'}_n\}$ the following inequality holds:*

$$\delta(G(D_n), G(D'_n)) \leq \frac{1}{n}\sum_{i=1}^{n}\delta(\mathbf{a}_i, \mathbf{a'}_i).$$

We can prove the existence of the geometric mean of more than three elements by a similar way as above. The geometric mean satisfies following properties which is proved by induction.

Theorem 3. *The geometric mean $G(\Delta_n)$ satisfies the permutation invariance and the left gyrotranslation invariance;*

$$G(\mathbf{x} \oplus_E \Delta_n) = \mathbf{x} \oplus_E G(\Delta_n),$$

where $\mathbf{x} \oplus_E \Delta_n = \{\mathbf{x} \oplus_E \mathbf{y} \ : \ \mathbf{y} \in \Delta_n\}$.

References

[1] A. A. Ungar, *The relativistic noncommutative nonassociative group of velocities and the Thomas rotation*, Result. Math. **16** (1989), 168–179

[2] A. A. Ungar, *Analytic Hyperboric Geometry and Albert Einstein's Special Theory of Relativity*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ. 2005. xviii+463 pp. ISBN: 981-256-457-8

[3] A. A. Unger, *A Gyrovector Space Approach to Hyperbolic Geometry*, Synthesis Lecture on Mathematics and Scatistics, 4. Morgan & Claypool Publishers, Williston, VT, 2009. xii+182 pp. ISBN: 978-1-59829-822-2

[4] R. Bhatia and J. Holbrook, *Riemannian geometry and matrix geometric means*, Linear Algebra Appl. **413** (2006), 594-618

[5] T. Ando, Chi-Kwong Li and R.Mathias, *Geometric means*, Linear Algebra Appl. **385** (2004), 305-334